EEE225 - Engineering Mathematics I (Differential Equations)
Week 4: First-Order Nonlinear ODEs

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Introduction

Nonlinear ODEs are typically difficult to solve analytically. Today, three techniques having a modest range of applicability will be presented:

1. Separating variables,
2. Transforming the nonlinear ODE into a linear one,
3. For almost linear systems, successive approximations.

Note that the great majority of the times, engineers solve nonlinear differential equations by numerical integration on digital computers.

Nonlinear Separable ODEs

The first method applies only to first-order separable equations; that is, to ODEs of the form

$$\frac{dy}{dt} = f(t)g(y), \quad y(t_0) = y_0 \tag{1}$$

To solve Eq. 4.1, separating variables can be applied by dividing through by $g(y)$:

$$\frac{1}{g(y)}\frac{dy}{dt} = f(t)$$

Then the shortcut procedure, which was introduced and justified in Week 3, can be employed:

$$\int_{y(t_0)}^{y} \frac{dy}{g(y)} = \int_{t_0}^{t} f(\tau)d\tau \tag{2}$$

Nonlinear Separable ODEs

Sometimes the integration on both sides of Eq. 4.2 can be performed and an implicit equation is obtained such that

$$G(y) = F(t) \tag{3}$$

In rare circumstances, we are then able to solve Eq. 4.3 for y as an explicit function of t. In the typical case when we cannot, we can still use Eq. 4.3 to determine the values of the function through graphical techniques. The nonlinear problems in this text have been selected so that they can be solved explicitly.

Example: Sounding Rocket Trajectory

As an example of the application of nonlinear equations in engineering, a sounding rocket that flies vertically to carry instruments into the upper atmosphere or edge of space for scientific measurements is considered. Roketsan operates a sounding rocket program as part of its activities.

Three fundamental questions flash about the sounding rocket's trajectory:

1. What is its maximum speed going up?
2. What altitude does it reach?
3. What is its maximum speed coming down?

The first prototype sounding rocket of Turkey

Q1. Maximum Speed Going Up (1/3)

The equation of motion for the thrusting phase (i.e., when the rocket motor is on) can be derived from the conservation of momentum and mass. The total rate of change of the momentum of the vehicle and the gases exiting from its engine (considered as a unit) is zero. The vehicle accelerates in reaction to the hot gases exiting it at high speed. The mass of the vehicle is reduced as the gases leave. The equation of motion is

$$m\frac{dV}{dt} + \frac{dm}{dt}c = 0$$

or

$$\frac{dV}{dt} = -\frac{1}{m}\frac{dm}{dt}c \tag{4}$$

where V is the speed of the vehicle, m is the mass, and c is the speed of the exit gases relative to the rocket.

Q1. Maximum Speed Going Up (2/3)

There are three variables in this equation. We can eliminate time by writing

$$\frac{dV}{dt} = \frac{dV}{dm}\frac{dm}{dt}$$

and then, cancelling dm/dt from both sides, Eq. 4.4 becomes the separable equation

$$\frac{dV}{dm} = -\frac{c}{m}$$

Following the procedure outlined in Eq. 4.1 through Eq. 4.3,

$$dV = -c\frac{dm}{m}$$

$$\int_0^{V(m)} dV = -c\int_{m_0}^{m}\frac{dm}{m}$$

Q1. Maximum Speed Going Up (3/3)

$$V(m) = c\ln(\frac{m_0}{m})$$

and in particular, at rocket burnout, that is, the moment when the rocket fuel is exhausted,

$$V_b = c\ln(\frac{m_0}{m_b}) \tag{5}$$

We will see shortly that V_b, the velocity at burnout, is the maximum speed the sounding rocket attains on its way up. Eq. 4.5 is the so-called rocket equation.

Q2. Maximum Altitude (1/7)

Newton's second law of motion is $F = ma$. After burnout and on the way up, the sounding rocket's trajectory is governed by the nonlinear differential equation

$$m_b\frac{dV}{dt} = -m_b - (\frac{1}{2}\rho S C_D)V^2 \tag{6}$$

Acceleration is dV/dt, the derivative of velocity with respect to time, and the forces are weight and air resistance (drag), both acting in the downward direction. Drag is proportional to the air density ρ, the rocket's cross-sectional area S, its drag coefficient C_D (at low speeds a constant typically around 0.02 or 0.03), and the square of the velocity.

Q2. Maximum Altitude (2/7)

We can rewrite Eq. 4.6 as

$$\frac{dV}{dt} = -g - kV^2 \tag{7}$$

where

$$k = \frac{\rho S C_D}{2m_b} \tag{8}$$

and then separate variables:

$$\frac{dV}{g + kV^2} = -dt \tag{9}$$

We want to make a substitution to change the left-hand side of Eq. 4.9 into the form $dv/(1 + v^2)$, which we know how to integrate.

Q2. Maximum Altitude (3/7)

We accomplish this by defining $v = V\sqrt{k/g}$. After a little algebra, Eq. 4.9 becomes

$$\frac{dV}{1 + v^2} = -\sqrt{gk}dt \tag{10}$$

Integrating the right-hand side of Eq. 4.10 from 0 to t, and the left-hand side correspondingly from $v_b = V_b\sqrt{k/g}$ to $v(t) = V(t)\sqrt{k/g}$:

$$\arctan(v(t)) - \arctan(v_b) = -\sqrt{gk}t \tag{11}$$

Solving Eq. 4.11

$$v(t) = \tan(\arctan(v_b) - \sqrt{gk}t)$$

Q2. Maximum Altitude (4/7)

$$V(t) = \sqrt{g/k}\tan\left(\arctan\left(\sqrt{k/g}V_b\right) - \sqrt{gk}t\right) \quad (12)$$

Eq. 4.12 tells us that the maximum speed on the way up is V_b. This is as we would expect from physical intuition; drag and gravity can only reduce the speed gained with the rocket engine. Now, the altitude $h(t)$ is given by

$$h(t) = \int_0^t V(\tau)d\tau = \int_0^t \sqrt{g/k}\tan\left(\arctan\left(\sqrt{k/g}V_b\right) - \sqrt{gk}\tau\right)d\tau \quad (13)$$

Eq. 4.13 may look formidable but it is not difficult to integrate. Let $t' = \sqrt{gk}\tau$ and

$$\arctan\left(\sqrt{k/g}V_b\right) = t'_a \quad (14)$$

Q2. Maximum Altitude (5/7)

Then Eq. 4.13 becomes

$$h(t) = \frac{1}{k}\int_0^{\sqrt{gk}t} \tan\left(t'_a - t'\right) = \frac{1}{k}\int_0^{\sqrt{gk}t} \frac{\sin\left(t'_a - t'\right)}{\cos\left(t'_a - t'\right)}dt'$$

$$= \frac{1}{k}\ln\left(\frac{\cos\left(t'_a - \sqrt{gk}t\right)}{\cos\left(t'_a\right)}\right) \quad (15)$$

The maximum altitude (apogee) is reached when $V(t) = 0$. From Eq. 4.12 and Eq. 4.14, this occurs when $\sqrt{gk}t = t'_a$. Then, from Eq. 4.15, the maximum altitude h_a is given by

$$h_a = \frac{1}{k}\ln\left(\frac{1}{\cos\left(t'_a\right)}\right) \quad (16)$$

Q2. Maximum Altitude (6/7)

From Eq. 4.14, $\tan(t'_a) = \sqrt{kV_b^2/g}$, so

$$h_d = \frac{1}{k}\ln\left(\frac{1}{\cos(t'_a)}\right) = \frac{1}{k}\ln(\sec(t'_a)) = \frac{1}{k}\ln\left(\sqrt{1 + kV_b^2/g}\right)$$

$$= \frac{1}{2k}\ln\left(1 + kV_b^2/g\right) \quad (17)$$

Summarising, we have found that the sounding rocket's maximum altitude is given by

$$h_a = \frac{1}{2k}\ln\left(1 + kV_b^2/g\right) \quad (18)$$

where $k = \rho S C_D/2m_b$.

Q2. Maximum Altitude (7/7)

As a check on this result, consider the case where the effect of drag is very small; that is, when k is small. Recalling that $\ln(1+x) \cong x$ when x is small, Eq. 4.18 becomes

$$h_a = \frac{1}{2k}\ln\left(1 + kV_b^2/g\right) \cong \frac{V_b^2}{2g} \quad (19)$$

which is the classic drag-free result.

Q3. Maximum Speed Coming Down (1/6)

When the sounding rocket is on the way down, its equation of motion is

$$m_b \frac{dV}{dt} = -m_b g + \left(\frac{1}{2}\rho S C_D\right) V^2 \quad (20)$$

With the same definitions and substitutions as in maximum altitude, Eq. 4.20 can be separated into the form

$$\frac{dv}{1-v^2} = -\sqrt{gk}dt \quad (21)$$

The left-hand side of Eq. 4.21, in contrast to that in Eq. 4.10, is not a familiar integrand. We proceed via a partial fraction expansion:

$$\frac{dv}{1-v^2} = \left(\frac{1}{1-v^2}\right) dv = \left(\frac{c_1}{1-v} + \frac{c_2}{1+v}\right) dv = -\sqrt{gk}dt \quad (22)$$

Q3. Maximum Speed Coming Down (2/6)

Several methods exist to find the coefficients c_1 and c_2. One way is to recombine over the common denominator:

$$\frac{c_1}{1-v} + \frac{c_2}{1+v} = \frac{1}{1-v^2}$$

$$\frac{c_1(1+v) + c_2(1-v)}{1-v^2} = \frac{(c_1+c_2) + (c_1-c_2)v}{1-v^2} = \frac{1}{1-v^2}$$

and then match coefficients of powers of v in the numerators. If two polynomials of the same order are everywhere equal then their coefficients are equal. The solutions are $c_1 = c_2 = 1/2$. Returning to Eq. 4.22, we have

$$\left(\frac{1/2}{1-v} + \frac{1/2}{1+v}\right) dv = -\sqrt{gk}dt = -dt' \quad (23)$$

Q3. Maximum Speed Coming Down (3/6)

Integrating the right-hand side of Eq. 4.23 from $t' = t'_a$ and the left-hand side correspondingly from $v = 0$,

$$-(1/2)\ln(1-v) + (1/2)\ln(1+v) = -(t' - t'_a)$$

$$\ln\left(\frac{1+v}{1-v}\right) = -2(t' - t'_a)$$

$$\frac{1+v}{1-v} = e^{-2(t'-t'_a)}$$

$$1 + v = (1-v)e^{-2(t'-t'_a)}$$

$$-v(1 + e^{-2(t'-t'_a)}) = 1 - e^{-2(t'-t'_a)}$$

$$v = -\frac{1 - e^{-2(t'-t'_a)}}{1 + e^{-2(t'-t'_a)}}$$

Q3. Maximum Speed Coming Down (4/6)

Multiplying numerator and denominator by $e^{t'-t'_a}$ yields a more favoured form

$$v = -\frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \quad (24)$$

where $\theta = t' - t'_a = \sqrt{gk}(t - t_a)$.
The right-hand side of Eq. 4.24 is the negative of a function called the hyperbolic tangent of θ, written as $\tanh(\theta)$. In dimensional form, from Eq. 4.24, the sounding rocket's velocity on the way down is given by

$$V = -\sqrt{g/k}\left(\frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}\right) = -\sqrt{g/k}\tanh(\theta) \quad (25)$$

where $\theta = \sqrt{gk}(t - t_a)$ and t_a is the time of apogee, the instant before the rocket begins to fall.

Q3. Maximum Speed Coming Down (5/6)

To determine the maximum speed, we first integrate Eq. 4.25 to obtain $h(t)$, find the time t_I when the rocket impacts the ground, and then use Eq. 4.25 to calculate the speed at that point.
The result is that, on its way down, the sounding rocket reaches its maximum speed just before it strikes the ground and its value is

$$V_m = \frac{-V_b}{\sqrt{1 + kV_b^2/g}} \tag{26}$$

As a check, note that for vanishing k, that is, for very low drag, Eq. 4.26 becomes

$$V_m = -V_b \tag{27}$$

as we expect from conservation of energy in the case where there is no atmosphere.

Q3. Maximum Speed Coming Down (6/6)

On the other hand, when $kV_b^2/g >> 1$, Eq. 4.26 becomes

$$V_m = -\sqrt{g/k} \tag{28}$$

In this case V_m is independent of V_b. The value $\sqrt{g/k}$ is called the terminal velocity.

Transforming the Nonlinear ODE into a Linear One

As it happens, the nonlinear equations for both upward and downward rocket flight (Eqs. 3.6 and 3.20) can be transformed into linear ones by changing the independent variable from t to h and the dependent variable from V to $V^2/2$. This shortcut method cannot determine the way velocity varies with time, but it can yield both the maximum altitude and the maximum speed on the way down.

It is good to know that the possibility of transforming nonlinear ODEs into linear ones exists but, unfortunately, such shortcuts are unavailable for most nonlinear problems and little can be said in general about when it is reasonable to look for one.

Exact Equations (1/9)

A differential expression $M(x, y)dx + N(x, y)dy$ is an exact differential in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$.

A first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an exact equation if the expression on the left side is an exact differential.

Criterion for an Exact Differential (2/9)

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$.

Then a necessary and sufficient condition that $M(x, y)dx+N(x, y)dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solving an Exact DE (3/9)

Solve $(e^{2y} - y \cos xy)dx + (2xe^{2y} - x \cos xy + 2y)dy = 0$.

Solution: The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy \sin xy - \cos xy = \frac{\partial N}{\partial x}$$

Start with the assumption that $\partial f/\partial y = N(x, y)$

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x \cos xy + 2y$$

$$f(x, y) = 2x \int e^{2y} dy - x \int cosxydy + 2 \int ydy + h(x)$$

Solving an Exact DE (4/9)

Remember, the reason x can come out in front of the symbol $\int$ is that in the integration with respect to y, x is treated as an ordinary constant. It follows that

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y \cos xy + h'(x) = e^{2y} - y \cos xy$$

and so $h'(x) = 0$ or $h(x) = c$.

Hence a family of solutions is

$$xe^{2y} - \sin xy + y^2 + c = 0$$

Integrating Factors (5/9)

Recall from the last week that the left-hand side of the linear equation $y' + p(t)y = g(t)$ can be transformed into a derivative when we multiply the equation by an integrating factor. The same basic idea sometimes works for a nonexact differential equation $M(x, y)dx + N(x, y)dy = 0$.

It is sometimes possible to find an integrating factor $\mu(x, y)$ so that after multiplying, the left-hand side of

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is an exact differential. In an attempt to find μ we turn to the criterion for exactness. The above equation is exact if and only if $(\mu M)_y = (\mu N)_x$, where the subscripts denote partial derivatives.

Integrating Factors (6/9)

By the product rule of differentiation the last equation is the same as $\mu M_y + \mu_y M = \mu N_x + \mu_x N$ or

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

If μ depends only on the variable y, then

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$$

In this case, if $(N_x - M_y)/M$ is a function of y, only then we can solve the above equation for μ.

Integrating Factors (7/9)

To sum up, the results for the DE

$$M(x,y)dx + N(x,y)dy = 0$$

- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for the DE is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for the DE is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

Transforming a Nonexact DE to an Exact DE (8/9)

The nonlinear first-order DE

$$xydx + (2x^2 + 3y^2 - 20)dy = 0$$

is not exact. We find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from $\mu(x)$ gets us nowhere since

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

depends on x and y. However $\mu(y)$ yields a quotient that depends only on y:

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}$$

Transforming a Nonexact DE to an Exact DE (9/9)

The integrating factor is then

$$e^{\int \frac{3}{y} dy} = e^{3 \ln y} = e^{\ln y^3} = y^3$$

After multiplying the given DE by $\mu(y) = y^3$, the resulting equation is

$$xy^4dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0$$

You are supposed to verify that the last equation is now exact and to show that a family of solutions is

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$$

Bernoulli's Equation (1/3)

The DE

$$\frac{dy}{dx} + p(x)y = f(x)y^n$$

where n is any real number, is called Bernoulli's equation and is named after the Swiss mathematician Jacob Bernoulli (1654 –1705).

Note that for $n = 0$ and $n = 1$, the above equation is linear. For $n \neq 0$ and $n \neq 1$, the substitution

$$u = y^{1-n}$$

reduces any nonlinear equation of the same form in the DE to a linear one.

Solving a Bernoulli DE (2/3)

Solve $x\frac{dy}{dx} + y = x^2y^2$.

Solution: Rewrite the given DE in the form of Bernoulli DE by dividing by x:

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2$$

With $n = 2$, substitute $y = u^{-1}$ via chain rule and

$$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$$

into the given equation and simplify. The result is

$$\frac{du}{dx} - \frac{1}{x} = -x$$

Solving a Bernoulli DE (3/3)

The integrating factor for this linear equation is

$$e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Integrating

$$\frac{d}{dx}[x^{-1}u] = -1$$

gives $x^{-1}u = -x + c$ or $u = -x^2 + cx$. Since $u = y^{-1}$, we have $y = 1/u$, and so a solution of the given equation is

$$y = \frac{1}{(-x^2 + cx)}$$

Successive Approximations for Almost Linear Systems

Consider the electric circuit shown in Figure 3.1.

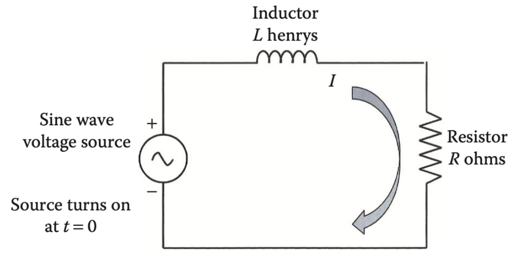

Figure 1: An RL electric circuit with a sine-wave voltage source

Successive Approximations for Almost Linear Systems

By Ohm's law, the voltage drop over the resistor in the direction of current is $V = IR$, the voltage drop over an inductor is $LdI/dt = (L/R)(dV/dt)$, and the voltage drop over the sine-wave source is $-V_0 \sin \omega t$.

Here, V_0 is the amplitude of the sine wave, in volts, and ω is the frequency of the oscillation in radians per second. Assume that the sine wave source turns on at $t = 0$ and that no current is flowing in the circuit at that time. Applying KVL to this circuit results in

$$L\frac{dI}{dt} + IR = V_0 \sin \omega t \tag{29}$$

Successive Approximations for Almost Linear Systems

Dividing through by L and defining $\lambda = R/L$ and $I_0 = V_0/R$, Eq. 4.29 can be written as

$$\frac{dI}{dt} + \lambda I = \lambda I_0 \sin \omega t \tag{30}$$

Suppose now that resistor in the circuit is slightly nonlinear. The resistance increase can occur when higher voltages raise the resistor's temperature. Then the circuit equation is modified to

$$\frac{dI}{dt} + \lambda I + \lambda \varepsilon I^3 = \lambda I_0 \sin \omega t \tag{31}$$

Nonlinear Eq. 4.31 cannot be solved in closed form in the same way that the linear Eq. 4.30 can.

Successive Approximations for Almost Linear Systems

However, if the nonlinear term εI^3 is not too large compared to I_0, that is, if the system modelled by Eq. 4.31 is almost linear, then we can employ a technique called successive approximations. We solve the nonlinear problem as a sequence of linear problems.

The first approximation ignores the nonlinearity altogether and is found by solving Eq. 4.30. Call that solution I_1. The next solution is obtained by approximating the nonlinear term as the known function of time $\lambda \varepsilon I_1^3$. Then we obtain I_2 through

$$\frac{dI_2}{dt} + \lambda I_2 = \lambda I_0 \sin wt - \lambda \varepsilon I_1^3 \tag{32}$$

$$I_2(t) = \int_0^t e^{-\lambda(t-\tau)}(\lambda I_0 \sin \omega\tau - \lambda \varepsilon I_1^3(\tau))d\tau \tag{33}$$

Successive Approximations for Almost Linear Systems

Recall Eqs. 2.17 and 2.34, now

$$\int_0^t e^{-\lambda(t-\tau)}\lambda I_0 \sin(\omega\tau)d\tau = I_1(t) \tag{34}$$

so, Eq. 4.33 becomes

$$I_2(t) = I_1(t) - \lambda\varepsilon \int_0^t e^{-\lambda(t-\tau)} I_1^3(\tau)d\tau \tag{35}$$

Proceeding in this way, we calculate $I_{n+1}(t)$ via

$$I_{n+1}(t) = I_1(t) - \lambda\varepsilon \int_0^t e^{-\lambda(t-\tau)} I_n^3(\tau)d\tau \tag{36}$$

Successive Approximations for Almost Linear Systems

Focusing on the steady-state solution only, performing the integral in Eq. 4.35 leads to

$$I_2(t) = G_1 I_0(\sin\omega t - \eta_2(t)) \quad (37)$$

where the effect of the nonlinearity is captured in $\eta_2(t)$:

$$\eta_2(t) = \frac{\alpha}{4}\left(3G_1\sin(\omega t - 2\theta_1) - G_3\sin(3\omega t - 3\theta_1 - \theta_3)\right) \quad (38)$$

and

$$G_n = \frac{\lambda}{\sqrt{\lambda^2 + (n\omega)^2}} \quad (39)$$

$$\theta_n = \arctan\left(\frac{n\omega}{\lambda}\right) \quad (40)$$

Successive Approximations for Almost Linear Systems

$$\alpha = \varepsilon\,(G_1 I_o)^2 \quad (41)$$

If we were to continue the sequence we would find

$$I_n(t) = I_1(t) - G_1 I_0 \eta_n(t) \quad (42)$$

where $\eta_n(t)$ is of the form

$$\eta_n(t) = \sum_{m=1}^{N_n} k_{mn}\sin(m\omega t - \phi_{mn}) \quad (43)$$

Eqs. 3.42 and 3.43 are specific examples of an important general fact.

Successive Approximations for Almost Linear Systems

When driven by a sine-wave source of frequency ω, stable and damped LTI systems respond in steady state with oscillations of that frequency only. Nonlinear systems, on the other hand, introduce multiples of that frequency, called harmonics. The phenomenon is called harmonic distortion and electrical engineers usually strive to avoid it by operating a system within the linear range of all of its analogue components.

It is easy to generalise the process described here to systems of higher order. However, the professional standard in engineering for solving nonlinear ODEs is numerical integration. Approximate analytical solutions can be valuable for providing insight and for checking for coding errors in differential equations to be solved by numerical integration.

Concluding Remarks (1/2)

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Concluding Remarks (2/2)

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